Non-zero solutions for matrices over $\mathbb{Z}_3$

Róbert Lukoťka

The task is to find an efficient algorithm that, given a matrix A over $\mathbb{Z}_3$, finds nowhere-zero vectors u and v such that $Au = v$. What is the complexity of the decision problem?

Facial packing number of plane graphs

Stanislav Jendroľ

A *facial packing number* $p_f(G)$ of a plane graph G is the smallest k such that there exists a colouring of the vertices of G with colours $1, \dots, k$ such that every facial path joining two vertices of the same colour i is of length at least $i + 1$ (consists of at least $i + 2$ vertices).
For a plane triangulation, four colours are enough, since in that case the facial packing is just a proper colouring. An upper bound for trees is known as well.
What is the smallest K such that $p_f(G) \leq K$ for every plane graph? Does such a constant exist?

Cycle-forest decomposition of cubic graphs

Raphael Steiner

Let G be a 3-connected cubic graph. Does there exist a decomposition of G into an induced cycle and a forest? Could planarity or cyclic 4-edge-connectivity help?

2-edge-colouring of bipartite graphs without monochromatic and alternating cycles

Bill Jackson

Problem Characterise the bipartite graphs which can be 2-edge-coloured in such a way that there are no monochromatic cycles and no alternating cycles.

Motivation The problem is motivated by a result of Daniel Bernstein [1]. We are given a bipartite graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^2$. The *completion matrix* of (G, p) is the $|E| \times 2|V|$ matrix with rows indexed by E and sequences of two consecutive columns indexed by V, in which the entries in the row corresponding to an edge $e = uv \in E$ are $p(v)$ and $p(u)$ in the columns indexed by u and v, respectively, and are zero elsewhere. Bernstein shows that, when p is generic, a set of rows F of $C(G, p)$ are linearly independent if and only if they induce a subgraph of G which has an orientation as described in the problem.

References

[1] D. I. Bernstein, Completion of tree metrics and rank-2 matrices, arXiv 1612.06797.

A general upper bound for the list chromatic number of triangle-free graphs

François Pirot

It is now well known through the work of Johanson, improved by Molloy, that the list chromatic number of triangle-free graphs of maximum degree Δ is at most $(1+o(1))\frac{\Delta}{\ln \Delta}$. We are interested in a similar upper bound depending only on the number of vertices of the graph, with no restriction on the maximum degree.
We can give such a bound for the chromatic number of triangle-free graphs, using the following lemma.

Lemma 1 (cf. Jensen and Toft) *Let $\mathcal{G}$ be a class of graphs closed by vertex deletion. Suppose for some $x_0 \geq 2$ that there is a continuous, non-decreasing function $f_{\mathcal{G}} : [0, \infty) \to \mathbb{R}^+$ such that every $G \in \mathcal{G}$ on $x \geq x_0$ vertices has a stable set of size at least $f_{\mathcal{G}}(x)$ vertices. Then every $G \in \mathcal{G}$ on $n \geq x_0$ vertices has chromatic number at most*

$$x_0 + \int_{x_0}^{n} \frac{dx}{f_{\mathcal{G}}(x)}.$$

As a corollary, using the result of Shearer that for any $\varepsilon > 0$, there exists $x_0 \geq 2$ such that the function $f_{\mathcal{G}}(x) = (1/\sqrt{2} - \varepsilon)\sqrt{x \ln x}$ satisfies the requirement of Lemma 1, we obtain that for any $\varepsilon > 0$, any triangle-free graph G on $n \geq n_\varepsilon$ vertices satisfies

$$\chi(G) \leq (2\sqrt{2} + \varepsilon)\sqrt{\frac{n}{\ln n}}.$$

Note that this is sharp up to some multiplicative constant, since the chromatic number of the triangle-free process is almost surely at least $(1/\sqrt{2} - \varepsilon)\sqrt{n/\ln n}$.

Question 1 *Is is true that for any triangle-free graph G on n vertices, $\chi_\ell(G) = O(\sqrt{n/\ln n})$?*

Using the fact that for any graph G, $\chi_\ell(G)/\chi(G) \leq \ln n$, we can ensure that $(2\sqrt{2} + \varepsilon)\sqrt{n \ln n}$ is an upper bound. Any better asymptotic bound is open.

Hamiltonicity of regular triangulations

Roman Nedela

Is it true that every k-valent triangulation on a surface is hamiltonian? (It is known to be true for $k \leq 6$, the problem is open for $k \geq 7$.)

3-coloring 4-regular graphs of high girth

František Kardoš

Does there exist a constant g such that every 4-regular graph with girth at least g is 3-colorable? (This is a last open case of a question posed by Grünbaum in the 60s.)

Homomorphisms between prisms

František Kardoš

Let G be a graph. We denote $G\square K_2$ the prism over G, the cartesian product of G with an edge. Is it true that if $G\square K_2 \to H\square K_2$, then $G \to H$?
It is obviously false if G is bipartite and $H = K_1$.

Update. Another counterexample was found by François Pirot and independently by Robert Šámal: It suffices to consider $H = W_5$ (a 5-wheel) and G being either $W_{2k+1}\square K_2$ or an odd wheel with the central vertex cloned.

Graph lightening

Théo Pierron

Let G be a connected undirected graph. Assume that each vertex contains a light that can be switched on or off. Consider someone walking through G. The switches are designed such that each time a vertex is visited, the state of its light changes (it becomes off if is was on, and on if is was off). Initially, no light is lit. The problem is the following: is it possible to find a walk in G such that every light is lit after the walk?
This problem can be solved by induction. Indeed, consider a spanning tree T of G, and an arbitrary initial vertex v_0. If T has a leaf f which is lit, then use induction on $G \setminus f$. Otherwise, walk from v_0 to f and then back to v_0, and apply induction on $G \setminus f$.
The first questions then arise: What is the length of a shortest lightening walk? Can it be computed in polynomial time?
Note that the main properties used in the induction is that the walk we construct can use edges in both ways, and have half-turns: they can cross an edge two times in a row. Natural restrictions then consist in either considering oriented graphs or forbidding such half-turns. It is then easy to find some graphs that cannot be lit. This leads to the new questions:

- Can we characterize which directed graphs can be lit? Or decide in polynomial time if it is the case?
- Can we compute the minimal number of people necessary to lit a given directed graph?
- Can we solve the two previous questions in the undirected case, but without half-turns?

A slight generalization of this problem would be to consider an initial configuration where not necessarily all the lights are switched off. It is then natural to ask the previous questions in this context, when the initial configuration is given as input.

Antisymmetric flows

Radek Hušek

The flow φ is antisymmetric if it is nowhere-zero and $\varphi(e)+\varphi(f) \neq 0$ for all edges e, f. Every bridgeless graph without directed 2-cut has an antisymmectric flow of a fixed size. The best upper bound is $\mathbb{Z}_2^3\mathbb{Z}_3^9$ [Dvořák, Kaiser, Králʼ, Sereni: A note on antisymmetric flows in graphs]. The best lower bound is 17, and moreover the graph showing this lower bound is planar [Marshall: On P-universal graphs]. The task is to reduce this very wide gap.

Decomposition of a doubled graph into two locally irregular multigraphs

Mariusz Woźniak

A multigraph is *locally irregular* if adjacent vertices have different degrees.
For a graph G let $G^{\|}$ be the multigraph obtained from G by replacing each edge by a pair of parallel edges.
Is it true that for every graph G the multigraph $G^{\|}$ can be edge-partitioned into two locally irregular multigraphs?